IOQM 2021-22 Part A

1. Three parallel lines L_1, L_2, L_3 are drawn in the plane such that the perpendicular distance between L_1 and L_2 is 3 and the perpendicular distance between L_2 and L_3 is also 3. A square $ABCD$ is constructed such that A lies on L_1, B lies on L_3 and C lies on L_2. Find the area of the square.

2. Ria writes down the numbers $1, 2, \ldots, 101$ in red and blue pens. The largest blue number is equal to the number of numbers written in blue and the smallest red number is equal to half the number of numbers written in red. How many numbers did Ria write with red pen?

3. Consider the set $\mathcal{T}$ of all triangles whose sides are distinct prime numbers which are also in arithmetic progression. Let $\triangle \in \mathcal{T}$ be the triangle with the least perimeter. If a° is the largest angle of $\triangle$ and if L is its perimeter, determine the value of $\frac{a}{L}$.

4. Consider the set of all 6-digit numbers consisting of only 3 digits, a, b, c, where a, b, c are distinct. Suppose the sum of all these numbers is 593999406. What is the largest remainder when the three digit number abc is divided by 100?

5. In parallelogram $ABCD$ the longer side is twice the shorter side. Let $XYZW$ be the quadrilateral formed by the internal bisectors of the angles of $ABCD$. If the area of $XYZW$ is 10, find the area of $ABCD$.

6. Let x, y, z be positive real numbers such that $x^2 + y^2 = 49$, $y^2 + yz + z^2 = 36$ and $x^2 + \sqrt{3}xz + z^2 = 25$. If the value of $2xy + \sqrt{3}yz + zx$ can be written as $p\sqrt{q}$ where p, q are integers and q is not divisible by square of any prime number, find $p + q$.

7. Find the number of maps $f : \{1, 2, 3\} \longrightarrow \{1, 2, 3, 4, 5\}$ such that $f(i) \le f(j)$ whenever $i < j$.

8. For any real number t, let $\lfloor t \rfloor$ denote the largest integer $\le t$. Suppose that N is the greatest integer such that
$$\left\lfloor \sqrt{\left\lfloor \sqrt{\left\lfloor \sqrt{N} \right\rfloor} \right\rfloor} \right\rfloor = 4$$
Find the sum of digits of N.

9. Let $P_0 = (3, 1)$ and define $P_{n+1} = (x_n, y_n)$ for $n \ge 0$ by
$$x_{n+1} = -\frac{3x_n - y_n}{2}, \qquad y_{n+1} = -\frac{x_n + y_n}{2}$$
Find the area of the quadrilateral formed by the points $P_{96}, P_{97}, P_{98}, P_{99}$.

10. Suppose that P is the polynomial of least degree with integer coefficients such that $P\left(\sqrt{7} + \sqrt{5}\right) = 2\left(\sqrt{7} - \sqrt{5}\right)$. Find $P(2)$.

11. In how many ways can four married couples sit in a merry-go-round with identical seats such that men and women occupy alternate seats and no husband seats next to his wife?

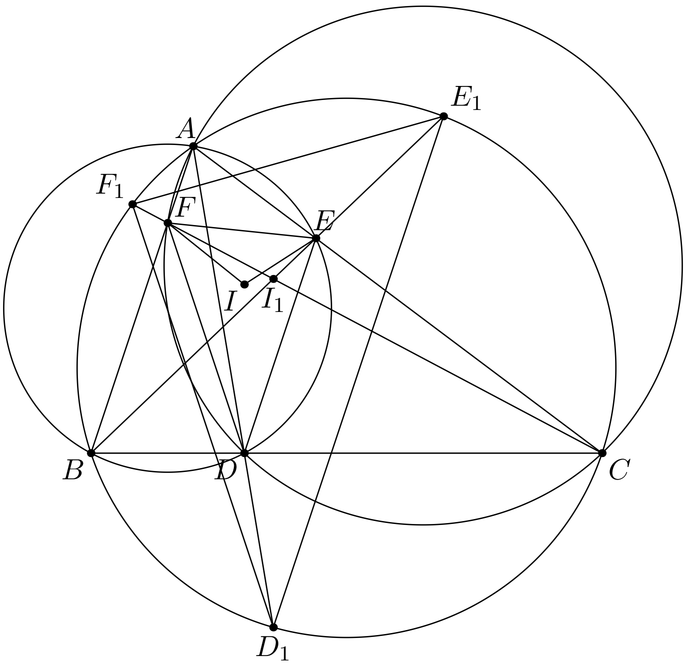
12. A 12×12 board is divided into 144 unit squares by drawing lines parallel to the sides. Two rooks placed on two unit squares are said to be non attacking if they are not in the same column or same row. Find the least number N such that if N rooks are placed on the unit squares, one rook per square, we can always find 7 rooks such that no two are attacking each other.

Question No.	Answer
1	45
2	68
3	08
4	98
5	40
6	30
7	35
8	24
9	08
10	40
11	12
12	73

IOQM 2022 Part B

Official Solutions

Problem 1. Let D be an interior point on the side BC of an acute-angled triangle ABC. Let the circumcircle of triangle ADB intersect AC again at $E(\neq A)$ and the circumcircle of triangle ADC intersect AB again at $F(\neq A)$. Let AD, BE and CF intersect the circumcircle of triangle ABC again at $D_1(\neq A)$, $E_1(\neq B)$ and $F_1(\neq C)$, respectively. Let I and I_1 be the incentres of triangles DEF and $D_1E_1F_1$, respectively. Prove that E, F, I, I_1 are concyclic.



Solution. Note that

$$\angle CF_1D_1 = \angle CAD_1 = \angle EAD = \angle EBD = \angle E_1BC = \angle E_1F_1C,$$

so F_1C is the bisector of $\angle D_1E_1F_1$. Similarly, E_1B is the bisector of $\angle D_1E_1F_1$, implying $I_1 = BE_1 \cap CF_1$. Now,

$$\begin{aligned}\angle EDF = \angle EDA + \angle FDA = \angle EBA + \angle FCA& \\ = \angle E_1BA + \angle F_1CA = \angle E_1D_1A + \angle F_1D_1A = \angle E_1D_1F_1&.\end{aligned}$$

Therefore

$$\angle EIF = 90^\circ + \frac{1}{2}\angle EDF = 90^\circ + \frac{1}{2}\angle E_1D_1F_1 = \angle E_1I_1F_1 = \angle EI_1F,$$

which proves the required concyclicity. $\square$

Problem 2. Find all natural numbers n for which there exists a permutation σ of $1, 2, \ldots, n$ such that

$$\sum_{i=1}^{n} \sigma(i)(-2)^{i-1} = 0.$$

Note: A *permutation* of $1, 2, \ldots, n$ is a bijective function from $\{1, 2, \ldots, n\}$ to itself.

Solution. Suppose that $n \equiv 1 \pmod 3$ and σ a permutation of $1, 2, \ldots, n$. Then

$$\sum_{i=1}^{n} \sigma(i)(-2)^{i-1} \equiv \sum_{i=1}^{n} \sigma(i) = \frac{n(n+1)}{2} \pmod 3,$$

and hence the left-hand side is non-zero.

We now show by induction that if $n \equiv 0$ or $2 \pmod 3$ then there exists a permutation of $1, 2, \ldots, n$ satisfying the given condition.

If $n = 2$ then the permutation given by $\sigma(1) = 2, \sigma(2) = 1$ satisfies the given condition. Similarly, if $n = 3$ then the permutation $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 1$ satisfies the given condition.

Suppose that for $n = m$ there exists a permutation σ satisfying the given condition. We consider the permutation τ of $1, 2, \ldots, m+3$ given by $\tau(1) = 2, \tau(2) = 3, \tau(m+3) = 1$ and $\tau(i) = \sigma(i-2) + 3$ for $i = 3, 4, \ldots, m+2$. Then

$$\begin{aligned}\sum_{i=1}^{m+3} \tau(i)(-2)^{i-1} &= 2 - 6 + (-2)^{m+2} + \sum_{i=3}^{m+2} 3 \cdot (-2)^{i-1} \\ &= 2 - 6 + (-2)^{m+2} - 4 \cdot ((-2)^m - 1) = 0\,.\end{aligned}$$

Thus, by induction it follows that for every $n \equiv 0$ or $2 \pmod 3$ there exists a permutation satisfying the given condition. $\square$

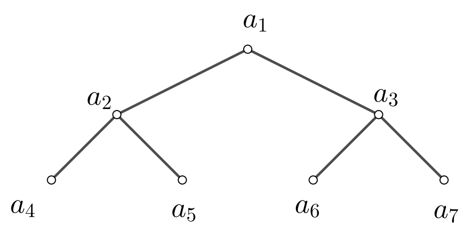
Problem 3. For a positive integer N, let $T(N)$ denote the number of arrangements of the integers $1, 2, \ldots, N$ into a sequence $a_1, a_2, \ldots, a_N$ such that $a_i > a_{2i}$ for all i, $1 \le i < 2i \le N$ and $a_i > a_{2i+1}$, for all i, $1 \le i < 2i+1 \le N$. For example, $T(3)$ is 2, since the possible arrangements are 321 and 312.

(a) Find $T(7)$.

(b) If K is the largest non-negative integer so that 2^K divides $T(2^n - 1)$, show that $K = 2^n - n - 1$.

(c) Find the largest non-negative integer K so that 2^K divides $T(2^n + 1)$.

Solution. (a) Given an arrangement $a_1, a_2, \ldots, a_7$, satisfying the given conditions, we can build a binary tree with nodes as in the Figure below. At each node, the root node



is greater than the child nodes. Conversely, any such tree gives a valid arrangement. Observing that the root of the tree must contain the maximum of the numbers, we can choose 3 out of the other 6 numbers in $\binom{6}{3}$ ways and build the left tree and the right tree, each in 2 ways and hence the number of such trees is $2 \cdot 2 \cdot \binom{6}{3} = 80$.

(b) Observe that $T(N)$ is also the number of ways of arranging any N distinct numbers into a sequence $a_1, a_2, \ldots, a_N$ satisfying the given conditions. Also, the given conditions imply that $a_1 =$ maximum of the numbers. Now, leaving out the maximum, the rest of the $2^n - 2$ numbers can be split into two groups of $2^{n-1} - 1$ numbers each and these can be individually put into a sequences $b_1, b_2 \ldots, b_{2^{n-1}-1}$ and $c_1, c_2, \ldots, c_{2^{n-1}-1}$ satisfying the conditions in $T(n-1)$ ways each. Now, the required arrangement of the original given sequence can be obtained as follows:

$$a_1, b_1, c_1, b_2, b_3, c_2, c_3, b_4, b_5, b_6, b_7, c_4, c_5, c_6, c_7, \ldots$$

This gives

$$T(2^n - 1) = T(2^{n-1} - 1)^2 \binom{2^n - 2}{2^{n-1} - 1} \tag{1}$$

We find the highest power of 2 that divides $\binom{2^n - 2}{2^{n-1} - 1}$:

We have

$$\begin{aligned} 2^{n-2}\binom{2^n}{2^{n-1}} &= 2^{n-2} \cdot \frac{2^n!}{2^{n-1}!2^{n-1}!} \\ &= 2^{n-2} \cdot \frac{2^n(2^n-1)(2^n-2)!}{2^{n-1}(2^{n-1}-1)!2^{n-1}(2^{n-1}-1)!} \\ &= (2^n - 1)\binom{2^n - 2}{2^{n-1} - 1} \end{aligned}$$

Now, the highest power of 2 that divides $\binom{2^n}{2^{n-1}}$ is

$$\left(2^{n-1} + 2^{n-2} + \cdots + 1\right) - 2\left(2^{n-2} + 2^{n-3} + \cdots + 1\right) = 1$$

Hence the highest power of 2 that divides $\binom{2^n - 2}{2^{n-1} - 1}$ is $n - 1$.

From the recurrence (1), if t_n is the highest power of 2 dividing $T(2^n - 1)$, then $t_n = 2t_{n-1} + n - 1$. From the initial conditions, $t_1 = 0, t_2 = 1, t_3 = 4$, we obtain, by an easy induction, that $t_n = 2^n - n - 1$.

(c) Suppose that $N = 2^n + 1$. It is easy to see that

$$T(2^n + 1) = T(2^{n-1} - 1)T(2^{n-1} + 1)\binom{2^n}{2^{n-1} + 1}$$

The highest power of 2 dividing $\binom{2^n}{2^{n-1} + 1}$ is n:

$$(2^{n-1} + 1)\binom{2^n}{2^{n-1} + 1} = \binom{2^n}{2^{n-1}} \cdot 2^{n-1}$$

Since the highest power of 2 dividing $\binom{2^n}{2^{n-1}}$ is 1, it follows that the highest power of 2 dividing $\binom{2^n}{2^{n-1} + 1}$ is n. Thus, if s_n denotes the highest power of 2 dividing $T(2^n + 1)$, then

$$s_n = s_{n-1} + 2^{n-1} - (n-1) - 1 + n = s_{n-1} + 2^{n-1}$$

Hence $s_n - s_1 = 2^n - 2$ and since $s_1 = 1$ (since $T(3) = 2$), it follows that the highest power of 2 dividing $T(2^n + 1)$ is $2^n - 1$. $\square$